Math 6000, Fall 2020 (Prof. Kinser), Homework 5

Nitesh Mathur

9 October 2020

Source Discussed problem/solutions with Zach Bryhtan and then went over drafts for the homework to get rid of erroneous writing.

Problem 1. *Skills developed: extending the concept of "exact sequence" to groups.* Let $1 \to H \xrightarrow{\alpha} G \xrightarrow{\beta} K \to 1$ be an *exact sequence* of groups, meaning that α and β are group homomorphisms such that:

(i) α is injective;

(ii) β is surjective;

(iii) im α = $\ker(\beta)$.

In particular, $K \simeq G/H$ (where H is identified with a subgroup of G via α.) Suppose that there exists a homomorphism $\beta' : K \to G$ such that $\beta \circ \beta' = 1_K$, the identity map on K (this is called a *splitting* of β).

Show that

(a) this determines a homomorphism $\phi : K \to Aut(H)$,

(b) giving an isomorphism $\theta : G \to H \rtimes_\varphi K$,

(c) such that the diagram below commutes.

$$\begin{array}{ccccccccc}
1 & \longrightarrow & H & \xrightarrow{\alpha} & G & \xrightarrow{\beta} & K & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle \theta} & & \downarrow{\scriptstyle id} & & \\
1 & \longrightarrow & H & \longrightarrow & H \rtimes_\varphi K & \longrightarrow & K & \longrightarrow & 1
\end{array}$$

(The maps on the bottom row are the standard inclusion and quotient for a semidriect product.)

Defs/Thms 1. A pair of morphism $X \xrightarrow{\alpha} Y \xrightarrow{B} Z$ is exact if im$(\alpha) = \ker(\beta)$.

2. A **short exact sequence** is an exact sequence of the form: $0 \to \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$

3. (From Class) Let H, N be groups. Recall,

$$H \times N = \{(h,n) | h \in H, n \in N\}$$

is a group via $(h_1, n_1).(h_2, n_2) = (h_1h_2, n_1n_2)$.

3b. Semidirect product is similar. The underlying set is the same but multiplication is "twisted" by choice of group homomorphisms $\phi : H \to Aut(N)$ in $H \rtimes_\phi N$ such that $(h_1, h_2).(h_2, n_2) = (h_1\phi(n_1)).h_2, n_1n_2)$

4. (D and F: 4.4 Proposition 13 - Pg. 135) Let H be a normal subgroup of G. Then G acts by conjugation on H as automorphisms of H. Specifically, the action of G on H by conjugation is defined for each $g \in G$ by

$$h \mapsto ghg^{-1}$$

for each $h \in H$.

5. Let $0 \to A \to B \to C \to 0$ and $0 \to A' \to B' \to C' \to 0$ be short exact sequences.

A morphism from the first sequence to the second sequence is a triple i.e. $\alpha : A \to A', \beta : B \to B', \gamma : C \to C'$ of $R-$module homomorphisms such that the diagram commutes.

Proof - Setup It is given that β is surjective $\Rightarrow \beta : G \to G/H = K$ is onto $\Rightarrow \beta(G) = K$.

By the First Isomorphism Theorem, since $\ker(\beta) \trianglelefteq G$, $G/\ker(\beta) \cong \beta(G) = K$.

Since we have an exact sequence of groups, $K \cong G/\ker(\beta) = G/\text{ im }(\alpha) = G/\alpha(H)$.

In particular, since $K \cong G/H, H = \alpha(H)$.

(a) $K \cong G/H \iff H \trianglelefteq G$.

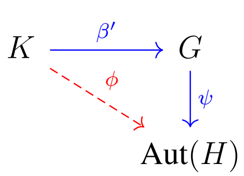
By Proposition 13 above, since H is normal subgroup, then G acts by conjugation on H as automorphisms of H. So, we can define the following map:

For each $g \in G$, define

$$\Psi : G \to \text{Aut}(H)$$
$$g \mapsto \tilde{\phi}_g = ghg^{-1}$$

for each $h \in H$.

Finally, we are given that there exists a $\beta' : K \to G$ such that $\beta \circ \beta' = 1_K$. Then, we have the following commutative diagram:



Hence, we have a homomorphism: $\phi : K \to \text{Aut}(H)$ defined by $\phi = \Psi \circ \beta'$.

(b) By Theorem 10, Let H and K be groups and let $\phi : K \to \text{Aut}(H)$ be a group homomorphism. Then, the operation is defined as follows:

$$(h_1, k_1)(h_2, k_2) = (h_1 k_1.h_2, k_1 k_2)$$

where (i) $H \trianglelefteq H \rtimes_\phi K$, (ii) $H \cap K = 1$, (iii) for all $h \in H, k \in K, hkh^{-1} = k.h = \phi(k)h$.

We need to adapt this in our situation. We have $K = G/H$, where $H \trianglelefteq G$. In particular, the quotient group has order $[G : H]$.

For $g \in G$, a left coset has the form $gH = \{gh | h \in H\}$ and right coset has the form $Hg = \{hg | h \in H\}$. (If H is a normal subgroup, then $gH = Hg$).

Show Show that $\theta : G \to H \rtimes_\phi K$ is an isomorphism.

Define the operation on $\theta : G \to H \rtimes_\phi G/H$ by:

$$\begin{aligned} G &\to H \rtimes_\phi G/H \\ a = gh &\mapsto (h, gH) \quad \text{for } g \in G, h \in H. \end{aligned}$$

The operation is well defined since the decomposition $a = gh$ is unique.

We need to show that θ is (i) one-to-one, (ii) onto, and (iii) a group homomorphism.

(b-i) We will show that the kernel is trivial.

$$\begin{aligned} \ker(\theta) &= \{a \in G | \theta(a) = (e_H, eH)\} \\ &= \{a \in G | \theta(gh) = (e_H, eH)\} \quad (\text{for } g \in G, h \in H) \\ &= \{a \in G | h = e_H, gH = eH\} \end{aligned}$$

Since $a = hg$, we have that $a = e_G$. Because the kernel is shown to be trivial, θ is injective.

Note (Credit Zach for spotting this)

If $a = gh, a, g \in G, h \in H$, then the above computation holds true only if $g \in H$ (since in line 3, we have $gH = eH = H$).

Suppose by way of contradiction, that $g \in H$ and $g \neq e_G$. Then, we have $a = g \cdot e_H \mapsto (e_H, eH) \Rightarrow g = e_G \Rightarrow a = e_G$ as well. (We will show below in (b-iii) that θ is a group homomorphism, so explicitly if $g \in H$ such that $g \neq e_G \Rightarrow g = h'$. Then, $a = gh = h'e_H = h' \Rightarrow \theta(a) = \theta(gh) = \theta(e_G \cdot h') = (h', eH) \neq (e_H, eH)$ by assumption, which is a contradiction. Hence, we need $g = e_G$).

(b-ii) Let (h_1, g_1H) be an arbitrary element of $H \rtimes_\phi G/H$. Then, we can choose $a_1 = g_1h_1 \in G$ such that $\theta(a_1) = \theta(g_1h_1) = (h_1, g_1H)$.

Since (h_1, g_1H) was arbitrary, we have shown that θ is surjective.

(b-iii) Let $a_1, a_2 \in G$, where $a_1 = g_1h_1, a_2 = g_2h_2$.

Then, by using the multiplication in semi-direct products as above, we get:

$$\begin{aligned}\theta(a_1)\theta(a_2) &= \theta(g_1h_1)\theta(g_2h_2)\\ &= (h_1, g_1H)\cdot(h_2, g_2H)\\ &= (h_1(g_1H).h_2, g_1Hg_2H) \quad \text{(by the action)}\\ &= (h_1g_1Hh_2(g_1H)^{-1}, g_1g_2H) \quad \text{(conjugation)}\\ &= \theta[(g_1h_1)(g_2h_2)]\\ &= \theta(a_1a_2)\end{aligned}$$

(c) (By 5. in Defs/Thms), If for two short exact sequences, we can show that there is a morphism from the first sequence to the second sequence via a triple, then then the diagram commutes.

It was given that the sequence $1 \to H \xrightarrow{\alpha} G \xrightarrow{\beta} K \to 1$, was an exact sequence of groups.

For the second sequence, $1 \to H \xrightarrow{i} H \rtimes_\phi K \xrightarrow{\pi_2} K \to 1$, it is clear that

(i) i is injective (since it is the standard inclusion)

(ii) π_2 is surjective (since it is the projection)

(iii) In particular, im $(i) = \ker(\pi_2)$ which can be seen from:

For $h \in H, i(h) = (h, 0) = \ker(\pi_2)$. Hence, this is also a short exact sequence.

Then, we have the following triple:

Let $\alpha : H \to H$ be the identity mapping on H.

Let $\theta : G \to H \rtimes_\phi K$ be the isomorphism defined in (b).

Let $\gamma : G/H \to G/H$ be the identity on $G/H = K$.

Since we have found a triple, the diagram commutes.

Problem 2. *Skills developed: practice with definitions below*. Prove that the following are equivalent for a ring R:

(i) every left $R-$module is projective, and (ii) every left $R-$module is injective.

Defs/Thms 1. A short exact sequence $0 \to A \to B \to C \to 0$ is called **split** if it is isomorphic to the sequence $0 \to A \xrightarrow{i_1} A \oplus C \xrightarrow{P_2} C \to 0$.

2. A $P \in R - \mathrm{Mod}$ is **projective** if $\mathrm{Hom}_R(P, -)$ is an exact functor. $Q \in R - \mathrm{Mod}$ is an **injective module** if $\mathrm{Hom}_R(-, Q)$ is an exact functor.

3a. A contravariant function F (between module categories) is **left exact** if $0 \to A \to B \to C \to 0$ exact $\Rightarrow 0 \to F(C) \to F(B) \to F(A)$ is exact.

3b. A covariant functor F between module categories is **right exact** if $0 \to A \to B \to C \to 0$ exact $\Rightarrow F(A) \to F(B) \to F(C) \to 0$.

3c. A functor which is both left and right exact (thus preserves short exact sequences) can also show it preserves all exact sequences) is called a **exact functor.**

4. **Proposition 30** Let P be an $R-$module. TFAE:

(i) P is projective.

(ii) For any $R-$modules L, M and N, if

$$0 \to L \xrightarrow{\Psi} M \xrightarrow{\varphi} N \to 0$$

is a short exact sequence, then

$$0 \to \mathrm{Hom}_R(P, L) \xrightarrow{\Psi'} \mathrm{Hom}_R(P, M) \xrightarrow{\varphi'} \mathrm{Hom}_R(P, N) \to 0$$

(iii) For any R-modules M and N, if $M \xrightarrow{\varphi} N \to 0$ is exact, then every $R-$module homomorphism from P into N **lifts** to an $R-$module homomorphism into M, i.e. given $f \in \mathrm{Hom}_R(P, N)$, there is a lift $F \in \mathrm{Hom}_R(P, M)$ making the diagram commute.

(iv) If P is a quotient of the $R-$module M then P is isomorphic to a direct summand of M, i.e. every short exact sequence $0 \to L \to M \to P \to 0$ splits.

(v) P is a direct summand of a free module i.e. $\exists$ set I and $P' \in R - \mathrm{Mod}$ such that $P \oplus P' \sim R^I$ (free module)

5. **Proposition 34** Let Q be an $R-$module. The FAE:

(i) Q is injective.

(ii) For any $R-$modules L, M, and N, if

$$0 \to L \xrightarrow{\Psi} M \xrightarrow{\phi} N \to 0$$

is a short exact sequence, then

$$0 \to \mathrm{Hom}_R(N, Q) \xrightarrow{\phi'} \mathrm{Hom}_R(M, Q) \xrightarrow{\Psi'} \mathrm{Hom}_R(L, Q) \to 0$$

is also a short exact sequence.

(iii) For any $R-$modules L and M, if $0 \to L \xrightarrow{\Psi} M$ is exact, then every $R-$ module homomorphism from L into Q lifts to an R-module homomorphism of M into Q i.e., given $f \in \mathrm{Hom}_R(L, Q)$ there is a lift $F \in \mathrm{Hom}_R(M, Q)$ making the following diagram commute: $0 \to L \xrightarrow{\Psi} M, L \xrightarrow{f} Q$, then there is an induced map $f : M \to Q$.

(iv) If Q is a submodule of the $R-$module M then Q is a direct summand of M, i.e. every short exact sequence $0 \to Q \to M \to N \to 0$ splits.

Show $\Rightarrow$ Show that a left $R-$ projective module is injective.

Suppose every left $R-$module is projective. Consider a short exact sequence:

$$0 \to L \to M \to P \to 0$$

Since P is projective, by Proposition 30 (iv), every short exact sequence splits, i.e. it is isomorphic to the sequence

$$0 \to L \xrightarrow{i} L \oplus P \xrightarrow{\pi_2} P \to 0$$

Hence L is precisely the injective module. Since we assumed that every $R-$module is projective, we are done.

(We can see this if we let $L = Q$. Then the statement above corresponds to Proposition 34 (iv), where Q is injective:)

$$0 \to Q \xrightarrow{i} Q \oplus P \xrightarrow{\pi_2} P \to 0$$

Show $\Leftarrow$ Show that a left $R-$ injective module is projective.

Suppose every left $R-$module is injective. Consider a short exact sequence:

$$0 \to Q \to M \to N \to 0$$

Since Q is injective, by Proposition 34 (iii), the sequence splits, i.e. it is isomorphic to the sequence.

$$0 \to Q \xrightarrow{i} Q \oplus N \xrightarrow{\pi_2} N \to 0$$

Hence, N is precisely the projective module. Since we assumed every left $R-$module is injective, we are done.

(We can see this if we let $N = P$. Then, the statement above corresponds to Proposition 30 (iv)):

$$0 \to Q \xrightarrow{i} Q \oplus P \xrightarrow{\pi_2} P \to 0$$

Problem 3. *Skills developed: practice with splitting and introduction to a useful module construction.*

This exercise introduces the concept of *pushout* to prove an equivalent condition for a module to be injective that was stated but not proved in class. given homomorphisms of $R-$modules $g_1 : M \to N_1$ and $g_2 : M \to N_2$, the *pushout* of f, g is the $R-$ module

$$N_1 \oplus_M N_2 := N_2 / \{(g_1(m), -g_2(m)) | m \in M\}.$$

The pushout fits into a commutative diagram:

$$\begin{array}{ccc} M & \xrightarrow{g_1} & N_1 \\ \downarrow{\scriptstyle g_2} & & \downarrow{\scriptstyle f_1} \\ N_2 & \xrightarrow{f_2} & N_1 \oplus_M N_2 \end{array}$$

where each f_i is the inclusion of the summand followed by the quotient.

(a) Prove that if g_1 is injective, then f_2 is injective.

(b) Let Q be an $R-$module such that every injective map $h : Q \to M$ splits. Prove that Q is injective. *Hint: use an appropriate pushout and part (a)*

Remark: There is a "dual" notion of pullback that can be used to prove directly the analagous characterization of projective modules, without going through the characterization that a projective module is a direct summand of a free module.

Proof (a) It is given that each f_i is the inclusion in the summand followed by the quotient, i.e.:

For $n_1 \in N_1$, $f_1(n_1) = (n_1, 0) + (g_1(m), -g_2(m))$, for $m \in M$. Similarly,

For $n_2 \in N_2$, $f_2(n_2) = (0, n_2) + (g_1(m), -g_2(m))$ for $m \in M$.

We are also given that g_1 is injective. Hence, for $m_1, m_2 \in M, g_1(m_1) = g_1(m_2) \Rightarrow m_1 = m_2$.

Show Let $n_{21}, n_{22} \in N_2$. If $f_2(n_{21}) = f_2(n_{22})$, show that $n_{21} = n_{22}$.

$$\begin{aligned} f_2(n_{21}) &= f_2(n_{22}) \\ (0, n_{21}) + (g_1(m), -g_2(m)) &= (0, n_{22}) + (g_1(m), -g_2(m)) \quad \text{(for } m \in M) \\ (0, n_{21}) - (0, n_{22}) + (g_1(m), -g_2(m)) &= (0, 0) \\ (g_1(m), n_{21} - n_{22} - g_2(m)) &= (0, 0) \end{aligned}$$

g_1 is injective $\iff \ker(g_1)$ is trivial $\Rightarrow g_1(m) = 0 \iff m = 0$.

Since M, N_2 are $R-$modules and $g_2 : M \to N_2$ is an $R-$module homomorphism, then 0 maps to $0 \Rightarrow g_2(m) = 0$.

Then, we have:

$$
\begin{aligned}
(0, n_{21} - n_{22} - g_2(m)) &= (0,0) \\
\Rightarrow (0, n_{21} - n_{22} - 0) &= (0,0) \\
\Rightarrow n_{21} - n_{22} &= 0 \\
\Rightarrow n_{21} &= n_{22}
\end{aligned}
$$

(b) Recall Proposition 34 (ii) states the following:

For any $R-$modules L and M if $0 \to L \xrightarrow{\Psi} M$ is exact, then every $R-$module homomorphism from L into Q lifts to an $R-$module homomorphism from L into Q lifts to an $R-$module homomorphism of M into Q, i.e. given $f \in Hom_R(L, Q)$, there is a lift $F \in \mathrm{Hom}_R(M, Q)$ making the diagram commute.

Note, that $0 \to L \xrightarrow{\Psi} M$ is exact $\iff$ Ψ is injective.

Since an *injective map* $h : Q \to M$ splits, for a short exact sequence, it is isomorphic to the following sequence:

$1 \to M \to M \oplus Q \to Q \to 1.$

Let L be an $R-$module. Let $g_1 : L \to M$ be an injective map and consider the following commutative diagram:

$$
\begin{array}{ccc}
L & \xrightarrow{g_1} & M \\
\downarrow{\scriptstyle g_2} & & \downarrow{\scriptstyle f_1} \\
Q & \xrightarrow{f_2} & M \oplus_L Q
\end{array}
$$

This is the pushout from part (a). Because $g_1 : L \to M$ is assumed to be injective, by part (a), f_2 is also injective.

Since $h : Q \to M$ splits, one of the equivalent definitions is that there exists an $R-$module homomorphism: $\pi_2 : M \oplus_L Q \to Q$.

Then, consider $F : M \to Q$ such that $F = f_1 \circ \pi_2$. Hence, we have found a lift such that the diagram commutes.

(The diagram commutes because $\pi_2 \circ f_1 \circ g_1 : L \to Q$ and similarly, $\pi_2 \circ f_2 \circ g_2 : L \to Q$. In particular, this is equal to $F \circ g_1 : L \to Q$, where F is the lift defined above).